Abstract

In [1], Hjorth proved that for every countable ordinal α, there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α that has models of all cardinalities less than or equal to $\aleph_\alpha$, but no models of cardinality $\aleph_{\alpha+1}$. Unfortunately, his solution yields not one $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α, but a set of $\mathcal{L}_{\omega_1,\omega}$-sentences, one of which is guaranteed to work.

The following is new: It is independent of the axioms of ZFC which of the Hjorth sentences works. More specifically, we isolate a diagonalization principle for functions from ω_1 to ω_1 which is a consequence of the *Bounded Proper Forcing Axiom* (BPFA) and then we use this principle to prove that Hjorth's solution to characterizing $\aleph_2$ in models of BPFA is different than in models of CH.

This raises the question whether Hjorth's result can be proved in an *absolute way* and what exactly this means, which we will discuss at the end of the talk.

This is joint work with Philipp Lücke.

References

Greg Hjorth.
Knight's model, its automorphism group, and characterizing the uncountable cardinals.
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Philipp Lücke, Ioannis Souldatos,
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Characterizing Cardinals by $\mathcal{L}_{\omega_1,\omega}$-sentences in an Absolute Way

Logic Colloquium 2022

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Preliminaries

1. An $\mathcal{L}_{\omega_1,\omega}$- sentence ϕ *characterizes* some cardinal κ, if ϕ has models in all cardinalities $[\aleph_0, \kappa]$ but no higher.
2. A countable model $\mathcal{M}$ characterizes some cardinal κ, if the same is true for its Scott Sentence.

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History of the Problem

In 2002, Hjorth proved the following:

Theorem

For all $\alpha < \omega_1$, there exists some complete $\mathcal{L}_{\omega_1,\omega}$- sentence ϕ_α which has models in all cardinalities $[\aleph_0, \aleph_\alpha]$ but no higher (ϕ_α characterizes $\aleph_\alpha$).

Remark: Hjorth's result is in ZFC and it is optimal.

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Comments

Since Hjorth there have been a few similar results.

Theorem (Baldwin, Koerwien, Laskowski(BKL))

For every $n \in \omega$, there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_n which characterizes $\aleph_n$.

However, Hjorth's construction is the only one known to work all $\aleph_\alpha$'s, $\alpha < \omega_1$.
So, for the rest of the talk we will focus on Hjorth's construction.

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Hjorth's Solution

- Unfortunately, Hjorth describes not one, but two constructions in his paper.
- Given some complete sentence ϕ which characterizes $\aleph_\alpha$, Hjorth's first construction yields a complete sentence which characterizes either $\aleph_\alpha$ or $\aleph_{\alpha+1}$.
- If the latter is the case, we are done.
- If not, then Hjorth introduces his second construction.
- If Hjorth's first construction characterizes $\aleph_\alpha$, then Hjorth's second construction characterizes $\aleph_{\alpha+1}$.
- Notice here that the failure of the first construction to characterize $\aleph_{\alpha+1}$ is used to prove that the second Hjorth construction does indeed characterize $\aleph_{\alpha+1}$.
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- Therefore, Hjorth's solution does not yield a single $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α, but a set of $\mathcal{L}_{\omega_1,\omega}$-sentences S_α, one of which is guaranteed to characterizes $\aleph_\alpha$.
- S_0 and S_1 are singletons.
- S_α is finite for finite α.
- For $\alpha = \omega$, iterating the first and the second construction ω-many times will yield a sentence that characterizes $\aleph_\omega$, regardless of what cardinal each iteration characterizes.
- So, S_ω is also a singleton.
- Similarly, S_λ is a singleton for all limit λ and S_α is finite for all $\alpha < \omega_1$.
- It was conjectured that it is independent of the axioms of ZFC which of the sentences in S_α characterizes $\aleph_\alpha$.
- New result: The conjecture is true.

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First Hjorth Construction

We briefly describe the first Hjorth construction.

Given: A countable model $\mathcal{M}$ which characterizes $\aleph_\alpha$.

There exists a countable structure F with the following properties:

1. F contains a complete countable graph G and (a copy of) $\mathcal{M}$.
2. Every edge of G is colored by an element of M. Denote by $C(a,b) = C(b,a)$ the color assigned to (a,b).
3. For $a, b \in G$, let $A^G(a,b) = \{c \in G | C(a,c) = C(b,c)\}$ (the set of agreements).
4. (Finite Agreement) For all $a, b \in G$, the set $A^G(a,b)$ is finite.

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5. (Finite Closure) For every X finite subset of G there exists some finite G_0, $X \subset G_0$ and G_0 is closed under A^G.
6. (Finite Extension) If G_0, G_1 are finite graphs with $G_0 \subseteq G$, $G_0 \subseteq G_1$ and G and G_1 introduce no new agreements to elements in G_0, then there exists an injection $i : G_1 \mapsto G$ with $i \restriction_{G_0} = id_{G_0}$ and $C^{G_1}(a,b) = C^G(i(a), i(b))$ for all $a, b \in G_1$.

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- Call any structure that satisfies the Scott sentence of F an M-full structure.

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Remark

The set M of colors is countable in the countable model, but may increase in other models (up to size $\aleph_\alpha$).

Theorem

1. *There exists an M-full structure size $\aleph_\alpha$.*
2. *Every M-full structure of size $\aleph_{\alpha+1}$ (if any) is maximal.*
3. *Therefore there is no M-full structure of size $\aleph_{\alpha+2}$.*

The crucial point is whether there exists a model of size $\aleph_{\alpha+1}$.

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The Case of $\aleph_2$

So, the first place where set theory may play a role is for $\alpha = 1$.

Question

For $\alpha = 1$ is there an M-full structure of size $\aleph_2$?

Lemma

If CH holds and M characterizes $\aleph_1$, then there is no M-full structure of size $\aleph_2$.

For the consistency of $\mathrm{ZFC}+$ "there is an *M*-full structure of size $\aleph_2$" we need more work.

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Property ($\triangle$)

We isolate a diagonalization property that we call ($\triangle$).

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1. Given a set X, we say that a map $m : [X]^{<\omega} \mapsto [X]^{<\omega}$ is *expansive* if $a \subseteq m(a)$ holds for every finite subset a of X.
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The importance of ($\triangle$) is apparent from the following theorem.

Theorem

Assume that ($\triangle$) *holds and let* M *be a countable model that characterizes* $\aleph_1$. *Then the countable* M*-full structure characterizes* $\aleph_2$.

Lemma

If (⊿) holds, then $2^{\aleph_0} > \aleph_1$.

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If (⊿) holds, then there exists a sequence $(A_\gamma | \gamma < \omega_2)$ *of unbounded subsets of* ω_1 *with the property that for all* $\delta < \gamma < \omega_2$, *the set* $A_\gamma \cap A_\delta$ *is finite.*

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If CH *holds and* G *is Add*(ω, ω_2)*-generic over* V, *then in* $\mathrm{V}[G]$ *there is no sequence* $(A_\gamma | \gamma < \omega_2)$ *of unbounded subsets of* ω_1 *with finite intersections.*

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Can we force $(\triangle)$*?*

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Yes!

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Definition

We let $\mathbb{D}$ denote the partial order defined by the following clauses:

1. A condition in $\mathbb{D}$ is a triple $p = \langle a_p, \mathscr{F}_p, \mathscr{X}_p \rangle$ such that the following statements hold:
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Theorem (Larson)

The partial order $\mathbb{D}$ is proper.

Forcing Axioms

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Given a partial ordering $\mathbb{P}$ and a cardinal κ, the Forcing Axiom $FA_\kappa(\mathbb{P})$ is the following statement:
For every collection $\{I_\alpha | \alpha < \kappa\}$ of maximal antichains of $\mathbb{P}$, there exists a filter G that intersects every I_α.
If Γ is a class of partial orderings, $FA_\kappa(\Gamma)$ is the statement that for every $\mathbb{P} \in \Gamma$, $FA_\kappa(\mathbb{P})$ holds.

Example

- Martin's Axiom MA_κ is $FA_\kappa(ccc)$, where $\kappa < 2^{\aleph_0}$.
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- Martin's Axiom MA_κ is $FA_\kappa(ccc)$, where $\kappa < 2^{\aleph_0}$
- Proper Forcing Axiom PFA is $FA_{\aleph_1}(proper)$

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Definition

Given a partial ordering $\mathbb{P}$ and a cardinal κ, the Forcing Axiom $FA_\kappa(\mathbb{P})$ is the following statement:
For every collection $\{I_\alpha | \alpha < \kappa\}$ of maximal antichains of $\mathbb{P}$, there exists a filter G that intersects every I_α.
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Bounded Forcing Axioms

Bounded forcing axioms are defined similarly, but the size of the antichains is now bounded.

Definition

Given a partial ordering $\mathbb{P}$ and a cardinal κ, the Bounded Forcing Axiom $BFA_\kappa(\mathbb{P})$ is the following statement:
$\mathbb{B} = r.o.(\mathbb{P}) \setminus \{0\}$, each of size at most κ, there exists a filter G that intersects every I_α.
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Generic Σ_1-Absoluteness

Definition

If Γ is a class of posets, $\Sigma_1(X)$-absoluteness for Γ is the following statement:

For every poset $\mathbb{P} \in \Gamma$, every Σ_1-formula $\phi(x_1, \ldots, x_n)$, and every $a_1, \ldots, a_n \in X$,

$$\phi(a_1, \ldots, a_n) \text{ iff } V^{r.o.(\mathbb{P})} \models \phi(\check{a}_1, \ldots, \check{a}_n)$$

(If a Σ_1 statement with parameters from X is forceable, then it is true.)

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Forcing Axioms and Generic Absoluteness

Forcing axioms are equivalent to generic Σ_1-absoluteness

Theorem

Let $\mathbb{P}$ be a partial ordering and κ an infinite cardinal of uncountable cofinality. Then the following are equivalent:

1. *$BFA_\kappa(\mathbb{P})$*
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3. *$\Sigma_1(\mathrm{H}(\kappa^+))$-absoluteness for $\mathbb{P}$.*

Corollary

The following statements are equivalent:

1. *BPFA holds.*
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BPFA *implies that* $(\triangle)$ *holds.*

Idea of the Proof Fix a sequence of functions $\vec{f} = (f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)$, a finite subset F of ω_1 and a expansive function $m : [\omega_1]^{<\omega} \mapsto [\omega_1]^{<\omega}$.

Let G be $\mathbb{D}$-generic over the ground model V. Work in $\mathrm{V}[G]$ and define $g = \bigcup \{a_p | p \in G\}$.

Then $g : \omega_1 \mapsto \omega_1$ with $F \cap range(g) = \emptyset$ and g satisfies the desired finite intersection property with all f_α's.

Since this statement can be formulate by a Σ_1-formula with parameters $\vec{f}, F, m \in \mathrm{H}(\omega_2)^{\mathrm{V}}$, we can use BPFA to conclude the given statement also holds in V.

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We can actually do better (i.e. reduce the consistency strength)

Theorem

$(\triangle)$ can be forced over a model of CH with a proper forcing $\mathbb{P}$ that satisfies the $\aleph_2$-chain condition.

Idea of the Proof The proper forcing $\mathbb{P}$ is a "matrix version" of Larson's forcing $\mathbb{D}$.

Summary:

- Hjorth proved that there exists a countable model M which characterizes $\aleph_1$ in all models of ZFC.
- Using M he constructed a countable M-full structure S.
- S characterizes $\aleph_1$ in models of CH and $\aleph_2$ in models of BPFA.
- One may ask if our results for $\aleph_2$ generalize to higher cardinalities, e.g. $\aleph_3$.
- To prove this one would have to extend our results for functions $f : \omega_1 \mapsto \omega_1$ to functions $f : \omega_2 \mapsto \omega_2$ (which is considerably harder).
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Question

1. *Can we have an absolute characterization of* $\aleph_\alpha$*,* $\alpha < \omega_1$*?*
2. *What does it mean to have an absolute characterization?*

Theorem

The characterization of $\aleph_n$*'s,* $n \in \omega$*, by Baldwin, Koerwien, and Laskowski is absolute.*

We suggest some answers for (2)

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1. *In L, there exists a unique code c for a complete $\mathcal{L}_{\alpha^+,\omega}$-sentence ψ_α such that $\Phi(\alpha, c)$ holds.*
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Fact (Shoenfield absoluteness)

Σ^1_3-statements are upwards absolute between transitive models of set theory with the same ordinals.

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Is there a Σ^1_3-formula $\Phi(v_0, v_1)$ in the language of second-order arithmetic with the property that the axioms of ZFC prove that the following statements hold:

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Theorem (Woodin)

The existence of a proper class of Woodin cardinals implies that the theory of $\mathrm{L}(\mathbb{R})$ with real parameters is generically absolute.

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The existence of a proper class of Woodin cardinals implies that the theory of $\mathrm{L}(\mathbb{R})$ with real parameters is generically absolute.

Question

Is there a formula $\Phi(v_0, v_1)$ in the language of set theory with the property that the theory
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1. *For every real a, there is a unique real b such that $\Phi(a, b)$ holds in $\mathrm{L}(\mathbb{R})$.*
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