Not too fast!

The Hanf Number for Joint Embedding

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This is a joint project with Will Boney.

Abstract Elementary Classes (AECs) are a general framework invented by Shelah that captures key properties of elementary classes and which maybe satisfied by non-elementary classes too.

AEC

A collection of models K is an AEC if it satisfies the following axioms. Let $A, B, C \in K$.

1. $A \prec B$ implies $A \subset B$;
2. $\prec$ is reflexive and transitive;
3. If $\langle A_i | i \in \kappa \rangle$ is an increasing $\prec$- chain, then
   (i) $\bigcup_{i \in I} A_i \in K$;
   (ii) for each $i \in I$, $A_i \prec \bigcup_{i \in I} A_i$; and
   (iii) if for each $i \in I$, $A_i \prec M$, then $\bigcup_{i \in I} A_i \prec M$.
4. If $A \prec B$, $B \prec C$ and $A \subset B$, then $A \prec B$.
5. There is a Lowenheim-Skolem number LS such that if $A \subset B$ and $B \in K$, then there is some $A' \in K$, $A \subset A' \prec B$, and $|A'| \leq |A| + LS$.

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Grossberg made the following conjecture.

Conjecture

For every λ, there is a cardinal $\mu(\lambda)$ such that for every Abstract Elementary Class (AEC) K, if K has the $\mu(LS(K))$-amalgamation property, then K has the λ-amalgamation property for all $\lambda \geq \mu(LS(K))$.

This cardinal $\mu(LS(K))$ (if it exists) is called the *Hanf number for amalgamation*.

Define similarly the *Hanf number for joint embedding*.

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Baldwin and Boney proved the existence of a Hanf number for joint embedding (and amalgamation), but their definition is different from Grossberg's.

Theorem (Baldwin,Boney)

If μ is a strongly compact cardinal, K is an AEC with $LS(K) < \mu$, and K satisfies jep/amalgamation cofinally below μ, then K satisfies jep/amalgamation in all cardinals $\geq \mu$.

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Grossberg's conjecture is still open, but at the moment we have two notions of a Hanf number for jep/amalgamation.

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Fact

Letting P be the property "there exists a model" and κ the Hanf number for P. The following are all equivalent.

1. *P holds in cardinality κ;*
2. *P holds in some cardinality above κ;*
3. *P holds in every cardinality above κ;*
4. *P holds in cofinally many cardinalities below κ;*
5. *P holds in eventually many cardinalities below κ; and*
6. *P holds in every cardinality (above LS)*

These equivalences rely heavily on the downward-closed property of model existence.
However, jep/amalgamation are not downward-closed.

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Definition

Fix a property P. A function $\mu_P(\kappa)$ from cardinals to cardinals will be called the *Hanf number for P* iff it satisfies the following:

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Theorem (Baldwin,Boney)

The $\mu_{jep}(\aleph_0)$ is bounded above by first (ω_1-) strongly compact.

Theorem (Boney, S.)

Then $\mu_{jep}(\aleph_0)$ is bounded below by the first measurable cardinal.

By results of Magidor, the first measurable and the first strongly compact can be consistently equal.
So, the known bounds are consistently optimal, but this is not always the case.

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The Construction

We define an AEC $(K, \prec_K)$.

The prototypical elements of K are be structures of the form

$$(\kappa, \mathcal{P}(\kappa), {}^{\omega}\mathcal{P}(\kappa), \in, \vee, \wedge, \cdot^{c}, \mathbf{1}, \cap, \pi_\alpha)_{\alpha<\omega},$$

where

1. ${}^{\omega}\mathcal{P}(\kappa)$ are the ω-sequences from $\mathcal{P}(\kappa)$ (the power set of κ);
2. $\in$ is the (extensional) 'member of' relation between κ and $\mathcal{P}(\kappa)$;
3. $(\mathcal{P}(\kappa), \vee, \wedge, \cdot^{c}, \mathbf{1})$ is the standard Boolean algebra;
4. $\cap : {}^{\omega}\mathcal{P}(\kappa) \to \mathcal{P}(\kappa)$ returns the intersection of all elements of the sequence; and
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The Construction

We define an AEC $(K, \prec_K)$.
The prototypical elements of K are be structures of the form

$$(\kappa, \mathcal{P}(\kappa), {}^{\omega}\mathcal{P}(\kappa), \in, \vee, \wedge, \cdot^c, \mathbf{1}, \cap, \pi_\alpha)_{\alpha<\omega},$$

where

1. ${}^{\omega}\mathcal{P}(\kappa)$ are the ω-sequences from $\mathcal{P}(\kappa)$ (the power set of κ);
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$(K, \prec_K)$ is defined by an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ.

1. There are 3 sorts: K for κ, P for $\mathcal{P}(\kappa)$, Q for ${}^{\omega}\mathcal{P}(\kappa)$
2. $(K, P, Q, \varepsilon, \vee, \wedge, \cdot^c, \mathbf{1}, \cap, \pi_\alpha)_{\alpha<\omega}$ satisfy the properties from the previous slide, except that we can not stipulate that P is the full powerset on K.
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Definition

Let $M \subset N \in \boldsymbol{K}$ and $X \in P^M$.

1. Define $\widehat{X}^M := \{x \in K^M \mid M \vDash x \varepsilon X\}$.
2. *M and N agree on X* iff $\widehat{X}^M = \widehat{X}^N$.
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The key factor for determining whether two $M, N \in \boldsymbol{K}$ can be jointly embedded is the size of K^M and K^N.

Lemma

There exists a model $M \in \boldsymbol{K}$ with $|M| = 2^\kappa$ and $|K^M| = \kappa$ such that for any other $N \in \boldsymbol{K}$ with $|K^N| = \kappa$ embeds into M.

Corollary

If $M_0, M_1 \in K$ and $|K^{M_0}| = |K^{M_1}|$, then M_0, M_1 can be jointly embeded to a larger model.

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Let $M \in \boldsymbol{K}$. M is *K-extendible* iff there is $N \in \boldsymbol{K}$ such that $M \prec_{\boldsymbol{K}} N$ and $K^M \subsetneq K^N$.

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Let $M_0, M_1 \in \boldsymbol{K}$. If $|K^{M_0}| < |K^{M_1}|$, then M_0 and M_1 can be jointly embedded if and only if M_0 is K-extendible.

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Let $M \in \mathbf{K}$. The following are equivalent:

1. *M is $\mathbf{K}$-extendible.*
2. *For all cardinals λ, there exists some $N \in \mathbf{K}$ with $M \prec_{\mathbf{K}} N$ and $|K^N \setminus K^M| = \lambda$.*
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Lemma

Let κ be less that the first measurable cardinal. Then there exists a model $M \in \mathbf{K}$ of size 2^κ that is not K-extendible.

M contains $(\kappa, \mathcal{P}(\kappa), {}^{\omega}\mathcal{P}(\kappa))$ and there is no $\aleph_1$-complete ultrafilter on $\mathcal{P}(\kappa)$.

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JEP(2^κ) fails, for every κ less that the first measurable.

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Recall that a cardinal κ is weakly compact if and only if for every κ-complete Boolean algebra $B \subset P(\kappa)$ generated by κ-many subsets, there is a κ-complete non-principal ultrafilter on B.

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A cardinal κ in *δ-weakly compact* for $\delta \leq \kappa$ iff every κ-complete Boolean algebra $B \subset P(\kappa)$ generated by κ-many subsets has a δ-complete non-principal ultrafilter on B.

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Theorem

If κ is a strong limit cardinal, then $\mathbf{K}$ satisfies JEP(κ).

Summarizing,

Theorem

Let μ be the first measurable.

1. *JEP($\aleph_0$) holds.*
2. *JEP(λ) fails for all $\aleph_1 \leq \lambda < \beth_\omega$.*
3. *If $\kappa < \mu$ and κ is a strong limit , then*
   (i) *JEP(κ) holds, but*
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Corollary

There is an AEC $(K, \prec_K)$ with $LS(K) = \aleph_0$ and

1. *JEP fails cofinally below the first measurable;*
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The Hanf number $\mu_{jep}(\aleph_0)$ is at least the first measurable cardinal.

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Theorem

Assume GCH. Given a club C on the first measurable μ, there is a generic extension $V[G]$ that preserves cardinalities and cofinalities, μ remains a measurable cardinal, and $\boldsymbol{K}$ satisfies JEP(λ) iff $\lambda \in \lim C$ or $\lambda \geq \mu$.

Theorem

*Let $\kappa \geq 2^{\aleph_1}$. Then **K** fails $AP(\kappa)$.*

Open Questions

(i) *Is there an AEC **K** that fails JEP eventually below κ, κ being the first measurable or the first super compact, but satisfies JEP in all cardinalities above κ?*

(ii) *Same question as (i), but satisfy JEP in one cardinality above κ?*

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- Thank you!
- Copy of these slides will be posted at `http://souldatosresearch.wordpress.com/`
- Questions?

- Thank you!
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