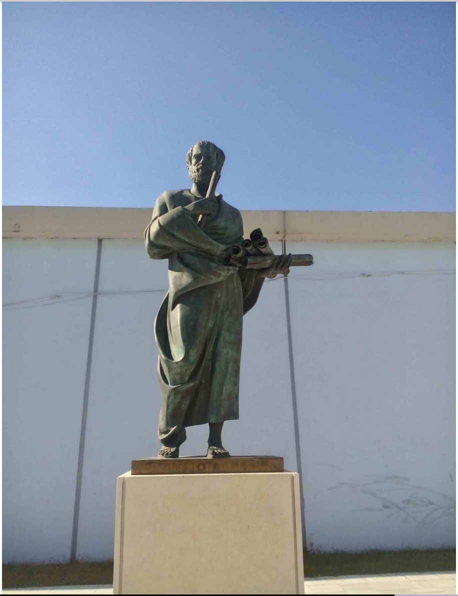

Statue of Aristotle at the Aristotle University of Thessaloniki, Greece

The model-existence and amalgamation spectra of $\mathcal{L}_{\omega_1,\omega}$-sentences

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Preliminaries

Definition

- $\mathcal{L}_{\omega_1,\omega}$ = first-order formulas $+ \bigvee_{n\in\omega} \phi_n + \bigwedge_{n\in\omega} \phi_n$
- $\mathcal{L}_{\omega_1,\omega}$ satisfies the Downward Lowenheim-Skolem Theorem, but fails the Upward Lowenheim-Skolem.

Spectra

Definition

For an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ, define the properties

- Model-existence at κ (ME(κ) for short) for "ϕ has a model of size κ"
- Amalgamation at κ (AP(κ) for short) for "ME(κ) + the models of ϕ of size κ satisfy amalgamation"
- The *model existence spectrum* of ϕ, ME-Spec(ϕ) $= \{\kappa | ME(\kappa)\}$
- The *amalgamation spectrum* of ϕ, AP-Spec(ϕ) $= \{\kappa | AP(\kappa)\}$

Main Question

What is known for the model existence and amalgamation spectra of $\mathcal{L}_{\omega_1,\omega}$-sentences?

Very Important Remark

The amalgamation notion is not unique. One needs to specify the embedding relation.

For the rest of the talk we assume that the embedding relation $\prec$ satisfies the following axioms (for an Abstract Elementary Class)

Assume A, B, C are models of some ϕ.

1. $A \prec B$ implies $A \subset B$;
2. $\prec$ is reflexive and transitive;
3. If $\langle A_i | i \in \kappa\rangle$ is an increasing $\prec$- chain, then
   (i) $\bigcup_{i\in I} A_i$ is a model of ϕ;
   (ii) for each $i \in I$, $A_i \prec \bigcup_{i\in I} A_i$; and
   (iii) if for each $i \in I$, $A_i \prec M$, then $\bigcup_{i\in I} A_i \prec M$.
4. If $A \prec B$, $B \prec C$ and $A \subset B$, then $A \prec B$.
5. There is a Lowenheim-Skolem number LS such that if $A \subset B$ and B satisfies ϕ, then there is some A' which satisfies ϕ, $A \subset A' \prec B$, and $|A'| \leq |A| + LS$.

Observation

- *By Downward Lowenheim-Skolem, ME-Spec(ϕ) is downward closed.*
- *So, either ME-Spec(ϕ)=$[\aleph_0, \aleph_\alpha]$ (right-closed spectrum) or ME-Spec(ϕ)=$[\aleph_0, \aleph_\alpha)$ (right-open spectrum)*

Definition

If ME-Spec(ϕ)=$[\aleph_0, \aleph_\alpha]$, then ϕ *characterizes* $\aleph_\alpha$.

Hanf Number

Theorem (Morley, López-Escobar)

Let ϕ be an $\mathcal{L}_{\omega_1,\omega}$-sentence. If ϕ has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1$, then it has models of all infinite cardinalities.

Reminder:

- $\beth_0 = \aleph_0$;
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$; and
- $\beth_\lambda = \sup_{\alpha<\lambda} \beth_\alpha$.

Corollary

If $\aleph_\alpha$ is characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence, then $\aleph_\alpha < \beth_{\omega_1}$.

Beths

Theorem (Malitz, Baumgartner)

For every $\alpha < \omega_1$, there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α that characterizes $\beth_\alpha$.

Thus, $\beth_{\omega_1}$ is optimal and is called the *Hanf number* for $\mathcal{L}_{\omega_1,\omega}$.

Alephs

Theorem (J.Knight)

There exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that characterizes $\aleph_1$.

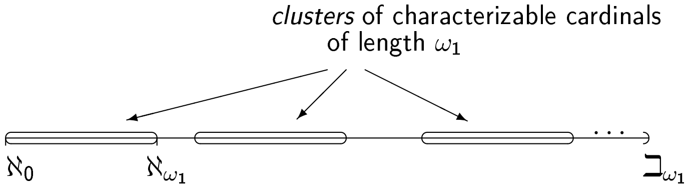
Theorem (Hjorth)

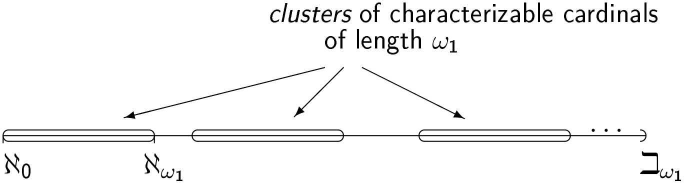
For every $\alpha < \omega_1$, there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α that characterizes $\aleph_\alpha$.

Hjorth's theorem generalizes to

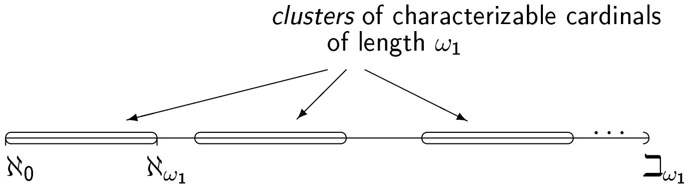
Theorem

Assume $\aleph_\beta$ is characterized by ϕ^β. Then for every $\alpha < \omega_1$, there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ^β_α that characterizes $\aleph_{\beta+\alpha}$.


clusters of characterizable cardinals of length ω_1
. . .
$\aleph_0$
$\aleph_{\omega_1}$
$\beth_{\omega_1}$



Under GCH there is only one cluster.



Under GCH there is only one cluster.
Under $\neg$GCH, it is consistent that there exist non-characterizable cardinals below $\beth_{\omega_1}$.

Theorem

The set of cardinals characterized by $\mathcal{L}_{\omega_1,\omega}$-sentences is closed under

1. *successor;*
2. *countable unions;*
3. *countable products;*
4. *powerset.*

Many of the results are true even for the set of cardinals characterized by **complete** $\mathcal{L}_{\omega_1,\omega}$-sentences.

Let C be the least set of cardinals that contains $\aleph_0$ and is closed under (1)-(4), e.g. successor, countable unions, countable products, and powerset.

Theorem (S.)

C is also closed for powers

Question (S.)

Does C contain all characterizable cardinals?

Consistently yes, e.g. under GCH.

Theorem (Sinapova,S.)

Consistently no.

- There exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ that characterizes the maximum of $2^{\aleph_0}$ and $\mathcal{B}$.

$$\mathcal{B} = \sup\{\kappa | \text{there exists a Kurepa tree with} \kappa \text{ many branches}\}.$$

- **Reminder**: A Kurepa tree has height ω_1, countable levels and more than $\aleph_1$ many branches.
So, $\aleph_1 < \mathcal{B} \leq 2^{\aleph_1}$.

Manipulating the size of Kurepa trees we can produce a variety of consistency results.

Theorem (Sinapova, S.)

If ZFC is consistent, then so are the following:

- *ZFC+* $(\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0})$
- *ZFC+* $(2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1})$ *+"*$\mathcal{B}$ *is a maximum", i.e. there exists a Kurepa tree of size* $\aleph_{\omega_1}$*".*

Assuming the consistency of ZFC+"a Mahlo exists", then the following is also consistent:

- *ZFC+* $(2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1})$ *+ "*$2^{\aleph_1}$ *is weakly inaccessible + "for every* $\kappa < 2^{\aleph_1}$ *there is a Kurepa tree with exactly* κ*-many maximal branches, but no Kurepa tree has exactly* $2^{\aleph_1}$*-many branches."*

Reminder: A cardinal λ is weakly inaccessible if it is a regular limit, i.e. $\lambda = \aleph_\lambda$.

Corollary

There is an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ for which it is consistent that

- *ME-Spec*$(\phi) = [\aleph_0, 2^{\aleph_0}]$;
- *CH (or $\neg$CH) + "$2^{\aleph_1}$ is a regular cardinal greater than $\aleph_2$" + "ME-Spec$(\phi) = [\aleph_0, 2^{\aleph_1}]$";*
- $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ + "*ME-Spec*$(\phi) = [\aleph_0, \aleph_{\omega_1}]$; *and*
- $2^{\aleph_0} < 2^{\aleph_1}$ + "$2^{\aleph_1}$ *is weakly inaccessible*" + "*ME-Spec*$(\phi) = [\aleph_0, 2^{\aleph_1})$.

Remarks

1. *Althought the characterization of $2^{\aleph_0}$ and $2^{\aleph_1}$ was known before, this is the first example that can consistently characterize either.*
2. *If $2^{\aleph_0} = \aleph_{\omega_1}$, then $\aleph_{\omega_1}$ is characterizable. This is the first example that consistently with $2^{\aleph_0} < \aleph_{\omega_1}$ characterizes $\aleph_{\omega_1}$. Moreover, the proof works for many other cardinals besides $\aleph_{\omega_1}$.*
3. *If $\kappa = \sup_{n\in\omega} \kappa_n$ and ϕ_n characterizes κ_n, then $\bigvee_n \phi_n$ has spectrum $[\aleph_0, \kappa)$. All previous examples of sentences with right-open spectra where of the form $\bigvee_n \phi_n$. This is the first example of a sentence with a right-open spectrum where the right endpoint has uncountable cofinality (indeed it is a regular cardinal).*

Recall that C is the least set that contains $\aleph_0$ and is closed under successor, countable unions, countable products, and powerset.

Corollary

If ZFC is consistent then so is ZFC+ $(2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1})$ + "$\mathcal{B}$ is not in the set C described above".

This refuted a "minimalist" view of characterizable cardinals.

Theorem

The set of cardinals characterized by $\mathcal{L}_{\omega_1,\omega}$-sentences is also closed under

5. *powers;*
6. $\kappa \mapsto Ded(\kappa) = \max\{\lambda |$ *there exists a linear order of size* λ *with a dense subset of size* $\kappa\}$;
7. $\kappa \mapsto \max\{\lambda |$ *there exists a* κ*-Kurepa tree with* λ *many branches*$\}$;
8. ...

We have only partial results for similar closures properties when *complete* $\mathcal{L}_{\omega_1,\omega}$-sentences are concerned.

Theorem (S.)

If $(\aleph_\alpha)^\kappa$ is characterized by a complete sentence, then the same is true for $(\aleph_{\alpha+\beta})^\kappa$, for all $\beta < \omega_1$.

Note: We do not assume that either $\aleph_\alpha$ or κ is characterizable.

Theorem (S.)

If $\aleph_\alpha$ and $\kappa^{\aleph_\alpha}$ are characterized by a complete sentence, then the same is true for $\kappa^{\aleph_{\alpha+\beta}}$, for all $\beta < \omega_1$.

Note: We do not assume that κ is characterizable.

Theorem (S.)

If $\aleph_\alpha$ and is characterized by a complete sentence, then the same is true for $2^{\aleph_{\alpha+\beta}}$, for all $0 < \beta < \omega_1$.

The question for $\beta = 0$ remains open. It is known under extra assumptions.

Negative Results

Theorem (S.)

*The set of characterizable cardinals is **not** closed under*

- *predecessor, and*
- *cofinality.*

Questions

1. *Do all these theorems and closure properties indicate a definability issue?*
2. *How to make this precise?*

Theorem (Morley, Barwise)

Let A be a countable admissible set with $o(A) = \gamma$, and let ϕ be a sentence of $\mathcal{L}_{\omega_1,\omega} \cap A$. Then either

- *ϕ characterizes some $\aleph_\alpha < \beth_\gamma$, or*
- *ϕ has arbitrarily large models.*

Question

What cardinals can be characterized by computable structures?

Theorem (S.Goncharov,J.Knight,S.)

(a) *Let $\mathcal{A}$ be a computable structure in a computable vocabulary τ, and let ϕ be a Scott sentence for $\mathcal{A}$. If ϕ has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1^{CK}$, then it has models of all infinite cardinalities.*

(b) *There exists a partial computable function I such that for each $a \in \mathcal{O}$, $I(a)$ is a tuple of computable indices for several objects, among which are a relational vocabulary τ_a and the atomic diagram of a τ_a-structure $\mathcal{A}_a$. The Scott sentence of $\mathcal{A}_a$ characterizes the cardinal $\beth_{|a|}$, where $|a|$ is the ordinal with notation a.*

Corollary

The Hanf number for Scott sentences of computable structures is $\beth_{\omega_1^{CK}}$.

Amalgamation Spectra

Remarks

- *Unlike model-existence, amalgamation spectra are not downwards closed.*
- *However, most of the known examples are either initial or co-initial segments of the model-existence spectra.*
- *We lack theorems of the form "If X is an amalgamation spectrum, then $Y(X)$ is also an amalgamation spectrum".*

The first example of an amalgamation spectrum that is not an interval is the following:

Theorem (J. Baldwin, M. Koerwien. C.Laskowski)

Let $1 \leq n < \omega$. There exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_n with ME-Spec$(\phi) = [\aleph_0, \aleph_n]$ and AP-Spec$(\phi) = [\aleph_0, \aleph_{n-2}] \cup \{\aleph_n\}$.

The reason that amalgamation holds in $\aleph_n$ is that all models are maximal.

The work on Kurepa trees yields the following results for amalgamation:

Theorem (Sinapova, S.)

There is an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ for which it is consistent that

- *AP-Spec*$(\phi) = [\aleph_1, 2^{\aleph_0}]$;
- *CH (or $\neg$CH) + "$2^{\aleph_1}$ is a regular cardinal greater than $\aleph_2$" + "AP-Spec$(\phi) = [\aleph_1, 2^{\aleph_1}]$";*
- $2^{\aleph_0} < \aleph_{\omega_1}$ *+ "AP-Spec$(\phi) = [\aleph_1, \aleph_{\omega_1}]$; and*
- $2^{\aleph_0} < 2^{\aleph_1}$ *+ "$2^{\aleph_1}$ is weakly inaccessible" + "AP-Spec$(\phi) = [\aleph_1, 2^{\aleph_1})$.*

Theorem (J. Baldwin, W. Boney)

Let ϕ be an $\mathcal{L}_{\omega_1,\omega}$-sentence, κ a strongly compact cardinal. If AP-Spec(ϕ) contains a cofinal subset of κ, then AP-Spec(ϕ) contains $[\kappa, \infty)$.

So, the first strongly compact is greater or equal to "the Hanf number for amalgamation" (this needs to be defined precisely).

Open Questions

- *Is it consistent that for an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ, AP-Spec(ϕ) is a cofinal subset of the first measurable?*
- *Can we find a characterization of the closure properties on the characterizable cardinals?*
- *Is it consistent that for an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ, ME-Spec(ϕ) equals $[\aleph_0, \aleph_{\omega_1})$ (right-open)? Same question for the AP-Spec(ϕ).*

References

- Thank you!
- Copy of these slides can be found at
  `http://souldatosresearch.wordpress.com/`
- Questions?